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A NOTE ON CERTAIN OSCILLATING SUMS

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A note on certain oscillating sums

by

A.E. Brouwer and J. van de Lune

ABSTRACT: Let  $S(N, \alpha) = \sum_{n=1}^N (-1)^{[n\alpha]}$ .

A characterization is given of all real  $\alpha$  for which  $S(N, \alpha) \geq 0$  for all  $N$ . In addition it is shown that the set consisting of all these  $\alpha$  has Lebesgue measure zero.

KEY WORDS & PHRASES: *exponential sums, continued fractions*



## 0. INTRODUCTION

In this note we investigate sums of the form

$$(0.1) \quad S_N(\alpha) = \sum_{n=1}^N (-1)^{[n\alpha]}, \quad (\alpha \in \mathbb{R}).$$

In particular we shall characterize the set  $P$  and the irrational elements of  $N$  where

$$(0.2) \quad P = \{\alpha \in \mathbb{R} \mid S_N(\alpha) \geq 0 \quad \text{for all } N \in \mathbb{N}\}$$

and

$$(0.3) \quad N = \{\alpha \in \mathbb{R} \mid S_N(\alpha) \leq 0 \quad \text{for all } N \in \mathbb{N}\}.$$

These characterizations (see theorem 2.1 and 4.1) will be given in terms of the regular continued fraction expansions of the corresponding  $\alpha$ .

In addition it will be shown that  $P$  and  $N$  have (Lebesgue) measure 0.

## 1. PREPARATIONS

We start dealing with  $P$ .

It is clear that

$$(1.1) \quad 0 \in P$$

and

$$(1.2) \quad \alpha \in P \iff \alpha + 2 \in P.$$

Hence, without loss of generality, we may assume that  $\alpha > 0$ . For the time being we also assume  $\alpha$  to be *irrational*.

A simple counting process reveals that if  $\alpha$  is positive then

$$(1.3) \quad S_N(\alpha) = \sum_{k=1}^M (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\} + (-1)^M \{N - [M\beta]\}$$

where  $M = [N\alpha]$  and  $\beta = \frac{1}{\alpha}$ .

Observe that for any  $M \in \mathbb{N}$

$$(1.4) \quad S_{[M\beta]}(\alpha) = \sum_{k=1}^M (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\}.$$

It is easily seen that (for positive  $\alpha$ )  $\alpha \in P$  if and only if

$$(1.5) \quad \sum_{k=1}^{2K} (-1)^{k-1} \{[k\beta] - [(k-1)\beta]\} \geq 0 \quad \text{for all } K \in \mathbb{N}.$$

Since  $2K$  is even (sic!) it follows that  $\alpha \in P$  if and only if for some  $z \in \mathbb{Z}$

$$(1.6) \quad \sum_{k=1}^{2K} (-1)^{k-1} \{[k(\beta+z)] - [(k-1)(\beta+z)]\} \geq 0 \quad \text{for all } K \in \mathbb{N}.$$

If we choose  $\beta + z > 0$  it follows that

$$(1.7) \quad \alpha \in P \iff \frac{1}{\beta+z} \in P.$$

In particular, taking  $z = -[\beta]$  we obtain

$$(1.8) \quad \alpha \in P \iff \frac{1}{\frac{1}{\alpha} - [\frac{1}{\alpha}]} \in P.$$

For any irrational  $\alpha$  with regular continued fraction expansion

$$\alpha = \langle a_0; a_1, a_2, a_3, \dots \rangle = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

we define

$$(1.9) \quad g(\alpha) = \langle a_2; a_3, a_4, a_5, \dots \rangle$$

and

$$(1.10) \quad \rho_k(\alpha) = \langle 0; a_k, a_{k+1}, a_{k+2}, \dots \rangle, \quad (k \in \mathbb{N}).$$

It is clear that

$$(1.11) \quad 0 < \rho_k(\alpha) < 1 \quad \text{for all } k \in \mathbb{N}$$

and

$$(1.12) \quad \frac{1}{\rho_k(\alpha)} = \langle a_k; a_{k+1}, a_{k+2}, \dots \rangle$$

so that

$$(1.13) \quad \frac{1}{\rho_k(\alpha)} = a_k + \rho_{k+1}(\alpha)$$

and

$$(1.14) \quad \left[ \frac{1}{\rho_k(\alpha)} \right] = a_k.$$

Hence

$$(1.15) \quad \frac{1}{\frac{1}{\rho_k(\alpha)} - \left[ \frac{1}{\rho_k(\alpha)} \right]} = \frac{1}{\rho_{k+1}(\alpha)} = a_{k+1} + \rho_{k+2}(\alpha).$$

LEMMA 1.1. *If  $\alpha$  is positive and irrational then*

$$(1.16) \quad \alpha \in P \iff (g(\alpha) \in P \text{ and } a_0 \equiv 0 \pmod{2}).$$

PROOF. ( $\Rightarrow$ ) Let  $\alpha \in P$ . Then  $a_0 \equiv 0 \pmod{2}$ . Indeed, if  $a_0 \not\equiv 0 \pmod{2}$  we would have  $S_1(\alpha) = -1$  so that  $\alpha \notin P$ .

Hence, by (1.2), it follows that  $\alpha - a_0 \in P$ , so that by (1.8) we have

$$(1.17) \quad \frac{1}{\frac{1}{\alpha - a_0} - \left[ \frac{1}{\alpha - a_0} \right]} \in P.$$

Since the left hand side of (1.17) equals  $g(\alpha)$  this part of the proof is complete.

( $\Leftarrow$ ). If  $g(\alpha) \in P$  then by (1.17) and (1.8) we have that  $\alpha - a_0 \in P$ . Since  $a_0 \equiv 0 \pmod{2}$  it follows from (1.2) that  $\alpha \in P$ .  $\square$

Define

$$(1.18) \quad P_N = \{0 \leq \alpha < 1 \mid S_n(\alpha) \geq 0 \text{ for all } n \leq N\}.$$

From this definition it is clear that

$$(1.19) \quad P_1 \supset P_2 \supset P_3 \supset \dots$$

and

$$(1.20) \quad P \cap [0, 1) = \bigcap_{N=1}^{\infty} P_N.$$

Let  $F_N$  be the Farey series of order  $N$ , restricted to the interval  $[0, 1)$ .

LEMMA 1.2.  $P_N$  is a (non-empty) union of a finite number of intervals of the form  $[a, b)$  with  $a < b$  where  $a$  and  $b$  are (rational) points of  $F_N$ .

PROOF. It is easily seen that

$$(1.21) \quad [0, \frac{1}{N}) \subset P_N \text{ and } [\frac{N-1}{N}, 1) \subset P_N$$

proving the "non-empty" part of the lemma.

Now let  $a$  and  $b$  be consecutive points of  $F_N$ . Then the proof is complete if we can show that

$$(1.22) \quad a \in P_N \Rightarrow [a, b) \subset P_N.$$



By definition,  $a \in P_N$  implies that

$$(1.23) \quad S_n(\alpha) = \sum_{k=1}^n (-1)^{[k\alpha]} \geq 0 \text{ for all } n \leq N.$$

Since for every fixed  $k \leq N$  the function  $[kx]$  is constant on each of the intervals  $[0, \frac{1}{k})$ ,  $[\frac{1}{k}, \frac{2}{k})$ ,  $\dots$ ,  $[\frac{k-1}{k}, 1)$  and since  $[a, b)$  is always contained in one of these intervals, the lemma follows from (1.23) by a right-continuity argument.  $\square$

COROLLARY 1.1. *P is left-closed. In other words:*

*If  $\{\alpha_n\}_{n=1}^{\infty}$  is a non-increasing sequence in P with limit  $\alpha$  then also  $\alpha \in P$ .*

COROLLARY 1.2. *P is (Lebesgue) measurable.*

LEMMA 1.3. *Let  $\alpha$  be irrational and positive.*

*If*

$$\beta = \frac{1}{\alpha}, \quad M \in \mathbb{N}, \quad N = [2M\beta], \quad z \in \mathbb{Z}, \quad \beta + z > 0, \quad K = [2M(\beta+z)]$$

*then*

$$(1.24) \quad S_N(\alpha) = S_K\left(\frac{1}{\beta+z}\right).$$

PROOF. This is a simple consequence of (1.4).  $\square$

If we choose  $z = -[\beta]$  in lemma 1.3 then

$$(1.25) \quad K = [2M(\beta - [\beta])] \leq [2M\beta] = N.$$

2. CHARACTERIZATION OF  $\mathcal{P}$ 

THEOREM 2.1. *Let  $\alpha$  be irrational and positive with regular continued fraction expansion*

$$\alpha = \langle a_0; a_1, a_2, a_3, \dots \rangle.$$

*Then*

$$(2.1) \quad \alpha \in \mathcal{P} \iff (a_{2i} \equiv 0 \pmod{2} \text{ for all } i \geq 0).$$

PROOF.  $(\Rightarrow)$ . Let  $\alpha \in \mathcal{P}$ . Then, by lemma 1.1 we have  $a_0 \equiv 0 \pmod{2}$  and  $g(\alpha) \in \mathcal{P}$ .

Observing that  $g(\alpha) = \langle a_2; a_3, a_4, \dots \rangle$  we must have  $a_2 \equiv 0 \pmod{2}$  etc.

$(\Leftarrow)$ . Now assume that  $a_{2i} \equiv 0 \pmod{2}$  for all  $i \geq 0$ . Suppose that  $\alpha \notin \mathcal{P}$ .

Then also  $\rho_1 \stackrel{\text{def}}{=} \alpha - a_0 \notin \mathcal{P}$ .

Hence

$$(2.2) \quad S_N(\rho_1) < 0 \text{ for some } N \in \mathbb{N}.$$

Choose  $N$  such that the inequality in (2.2) holds true and such that  $N$  is *minimal*. Since  $0 < \rho_1 < 1$  we may consider the position of  $\rho_1$  with respect to the Farey series of order  $N$ .

For every  $n \in \mathbb{N}$  such that  $n \leq N$ , the function  $[nx]$  is constant on the canonical (= smallest) intervals of the form  $[a, b)$  corresponding to  $F_N$ .

Hence, since  $\rho_1$  is irrational, there exists an open interval  $I$  containing  $\rho_1$  such that

$$(2.3) \quad S_N(\gamma) < 0 \text{ for all } \gamma \in I.$$

Because of the minimality of  $N$  there exists an  $M \in \mathbb{N}$  such that  $N = \left\lceil \frac{2M}{\rho_1} \right\rceil$ .

From continuity arguments concerning regular continued fractions it follows that there exists an  $\ell \in \mathbb{N}$  such that all irrational numbers  $x > 0$  defined by

$$(2.4) \quad x = \langle 0; a_1, a_2, \dots, a_{2\ell-1}, a_{2\ell}, m_{2\ell+1}, m_{2\ell+2}, \dots \rangle$$

(with  $m_j \in \mathbb{N}$ ,  $j \geq 2\ell+1$ )

are such that

$$(2.5) \quad x \in I \quad \text{and} \quad \left\lfloor \frac{2M}{x} \right\rfloor = \left\lfloor \frac{2M}{\rho_1} \right\rfloor = N.$$

Observe that (since  $a_{2i} \equiv 0 \pmod{2}$ )

$$(2.9) \quad S_{N_1} \left( \frac{1}{\frac{1}{x_0} - \left\lfloor \frac{1}{x_0} \right\rfloor} \right) = S_{N_1} (< a_2; a_3, a_4, \dots, a_{2\ell}, N, 1, 1, 1, \dots >) =$$

$$= S_{N_1} (< 0; a_3, a_4, \dots, a_{2\ell}, N, 1, 1, 1, \dots >) = S_{N_1} (\rho_3(\alpha)).$$

Without loss of generality we may assume that  $N_1$  is the smallest natural number for which

$$S_{N_1} (\rho_3(\alpha)) < 0.$$

Continuing this reduction we will ultimately find a natural number  $N_\ell$  such that

$$(2.10) \quad S_{N_\ell} (< 0; N, 1, 1, 1, \dots >) < 0 \text{ with } N_\ell \leq N.$$

On the other hand, since

$$(2.11) \quad < 0; N, 1, 1, 1, \dots > = \frac{1}{N+\delta} (< \frac{1}{N} >)$$

for some  $\delta > 0$  and since  $N_\ell \leq N$  we have

$$(2.12) \quad S_{N_\ell} (< 0; N, 1, 1, 1, \dots >) > 0.$$

Since this contradicts (2.10) the proof is complete.

THEOREM 2.2. *If  $\alpha$  is rational then  $\alpha \in P$  if and only if the canonical continued fraction expansion of  $\alpha$  is of the form*

$$(2.13a) \quad \alpha = \langle a_0; a_1, a_2, \dots, a_{2\ell-1}, a_{2\ell} \rangle$$

with

$$(2.13b) \quad \alpha_{2i} \equiv 0 \pmod{2} \text{ for all } 0 \leq i \leq \ell.$$

PROOF. Suppose  $\alpha$  satisfies (2.13).

Then

$$(2.14) \quad \alpha < \alpha_N \text{ for all } N \in \mathbb{N}$$

where

$$(2.15) \quad \alpha_N = \langle a_0; a_1, a_2, \dots, a_{2\ell-1}, a_{2\ell}, 2N, 2N, 2N, \dots \rangle.$$

Observing that  $P$  is left closed and that

$$(2.16) \quad \lim_{N \rightarrow \infty} \alpha_N = \alpha \text{ and } \alpha_N \in P.$$

it follows from (2.14) that  $\alpha \in P$ .

Now assume that (2.13) is not satisfied. Observe that if  $\alpha$  is positive and rational then (compare (1.3))

$$(2.16) \quad S_N(\alpha) = \lim_{\epsilon \downarrow 0} \left\{ \sum_{k=1}^M (-1)^k \{ [k(\beta-\epsilon)] - [(k-1)(\beta-\epsilon)] \} + \right. \\ \left. + (-1)^M \{ N - [M(\beta-\epsilon)] \} \right\}$$

where

$$M = \lim_{\epsilon \downarrow 0} [N(\alpha+\epsilon)] = [N\alpha].$$

From this we obtain that (for positive  $\alpha$ )  $\alpha \in P$  if and only if for all  $K \in \mathbb{N}$

$$(2.17) \quad \lim_{\varepsilon \downarrow 0} \sum_{k=1}^{2K} (-1)^{k-1} \{[k(\beta-\varepsilon)] - [(k-1)(\beta-\varepsilon)]\} \geq 0$$

so that, similarly as in section 1, for  $\alpha > 0$  and  $\alpha \in Q$  we have

$$(2.18) \quad \alpha \in P \iff \left(\frac{1}{\beta+z} \in P \text{ for some } z \in \mathbb{Z}\right).$$

In particular we use (2.18) with  $z = -[\beta]$ .

CASE 1.  $\alpha = \langle a_0; a_1, a_2, \dots, a_{2\ell-1} \rangle$

Assuming that  $\alpha \in P$  we would ultimately obtain that  $\langle a_{2\ell-2}; a_{2\ell-1} \rangle \in P$  so that we must have  $a_{2\ell-2} \equiv 0 \pmod{2}$  and hence

$$(2.19) \quad \langle 0; a_{2\ell-1} \rangle = \frac{1}{a_{2\ell-1}} \in P.$$

However, it is easily verified that  $P$  does *not* contain any of the numbers  $\frac{1}{n}$ ,  $n \in \mathbb{N}$ .

CASE 2.  $\alpha = \langle a_0; a_1, a_2, \dots, a_{2\ell-1}, a_{2\ell} \rangle$

with  $a_{2i} \not\equiv 0 \pmod{2}$  for some  $i$ .

Repeated use of (2.18) reveals that  $\alpha \notin P$ .  $\square$

### 3. THE MEASURE OF THE SET $P$ .

THEOREM 3.1. *The set  $P$  has measure 0.*

PROOF. Define  $P^* = \{P \setminus Q\} \cap [0, 1)$ .

Let  $(a_i, b_i)$  be some countable system of open intervals such that  $0 \leq a_i < b_i$  for all  $i$  and

$$(3.1) \quad E \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} (a_i, b_i) \supset P^*.$$

From the characterization of the irrational points belonging to  $P$  it is clear that

$$(3.2) \quad P^* = \bigcup_{\substack{k, a=1 \\ x \in P^*}}^{\infty} \left\{ \frac{1}{k + \frac{1}{2a+x}} \right\}$$

so that

$$(3.3) \quad P^* \subset \bigcup_{\substack{k, a=1 \\ x \in E}}^{\infty} \left\{ \frac{2a+x}{k(2a+x)+1} \right\}.$$

Since for all fixed  $k, a \in \mathbb{N}$  the function

$$(3.4) \quad \frac{2a+x}{k(2a+x)+1}, \quad (x>0)$$

is increasing we obtain that ( $\lambda$  denoting Lebesgue measure)

$$\begin{aligned} (3.5) \quad \lambda(P^*) &\leq \sum_{i,k,a=1}^{\infty} \left\{ \frac{2a+b_i}{k(2a+b_i)+1} - \frac{2a+a_i}{k(2a+a_i)+1} \right\} = \\ &= \sum_{i,k,a=1}^{\infty} \frac{b_i - a_i}{\{k(2a+b_i)+1\}\{k(2a+a_i)+1\}} \leq \\ &\leq \sum_{i,k,a=1}^{\infty} \frac{b_i - a_i}{4k^2 a^2} = \frac{1}{4} \left( \frac{\pi^2}{6} \right)^2 \cdot \lambda(E). \end{aligned}$$

It follows that

$$(3.6) \quad \lambda(P^*) \leq \frac{\pi^4}{144} \lambda(E) < \frac{7}{10} \lambda(E).$$

Since  $P^*$  is measurable and  $E$  may be chosen such that

$$(3.7) \quad \lambda(E) < \lambda(P^*) + \varepsilon,$$

it follows easily that we must have

$$(3.8) \quad \lambda(P^*) = 0$$

and hence

$$(3.9) \quad \lambda(P) = 0.$$

#### 4. THE SET $N$

THEOREM 4.1. *If  $\alpha$  is irrational then*

$$(4.1) \quad \alpha \in N \iff -\alpha \in P.$$

PROOF. Observe that

$$(4.2) \quad [x] + [-x] = -1 \quad \text{for all } x \in \mathbb{R} \setminus \mathbb{Z}.$$

Hence, if  $\alpha$  is irrational then

$$\begin{aligned} (4.3) \quad S_N(\alpha) &= \sum_{n=1}^N (-1)^{[n\alpha]} = \sum_{n=1}^N (-1)^{-1-[-n\alpha]} = \\ &= - \sum_{n=1}^N (-1)^{-[-n\alpha]} = - \sum_{n=1}^N (-1)^{[n(-\alpha)]} \end{aligned}$$

so that

$$(4.4) \quad S_N(\alpha) \leq 0 \iff S_N(-\alpha) \geq 0,$$

proving the theorem.

REMARK. In general, formula (4.1) does not hold true for  $\alpha \in \mathbb{Q}$  as may be seen from the following example:  $1 \in \mathbb{N}$ ,  $-1 \notin P$ .

COROLLARY. *The set  $N$  has measure zero.*

## 5. ONE MORE PROPERTY OF $P$ (resp. $N$ )

THEOREM 5.1. *For every irrational  $\alpha \in P$  we have that*

$$(5.1) \quad S_N(\alpha) = 0 \text{ for infinitely many } N \in \mathbb{N}.$$

In order to prove this we use the following

LEMMA 5.1. *If the positive integers  $p$  and  $q$  are such that  $p$  is even and  $(p, q) = 1$  then*

$$(5.2) \quad S_{q-1} \left( \frac{p}{q} \right) = 0.$$

PROOF. Consider the  $q-1$  numbers

$$\frac{p}{q}, \frac{2p}{q}, \dots, \frac{(q-1)p}{q}.$$

Since  $(p, q) = 1$  none of these numbers is an integer and since  $p$  is even  $q$  is odd so that  $q-1$  is even.

Since  $p$  is even we have for  $1 \leq r \leq \frac{q-1}{2}$  that the integers

$$\left[ r \frac{p}{q} \right] \text{ and } \left[ (q-r) \cdot \frac{p}{q} \right]$$

have different parity from which it is clear that  $S_{q-1} \left( \frac{p}{q} \right) = 0$ .

PROOF OF THEOREM 5.1.

Without loss of generality, we may assume that  $0 < \alpha < 1$ .



Let  $\alpha = \langle 0; a_1, a_2, \dots \rangle$  and let

$$\frac{A_0}{B_0} = \frac{0}{1}, \frac{A_1}{B_1} = \frac{1}{a_1}, \frac{A_2}{B_2} = \frac{a_2}{a_1 a_2 + 1}, \dots, \frac{A_n}{B_n}, \dots$$

be the corresponding convergents.

Since  $\alpha \in \mathcal{P}$  we have that  $a_{2i} \equiv 0 \pmod{2}$  for all  $i \geq 1$  from which it is easily seen that  $A_{2n}$  is even for all  $n$ .

In order to prove the theorem it suffices to show that for all  $n \in \mathbb{N}$

$$(5.3) \quad \sum_{k=1}^{B_{2n}-1} (-1)^{[k\alpha]} = 0.$$

Since  $A_{2n}$  is always even it follows from lemma 5.1 that

$$(5.4) \quad \sum_{k=1}^{B_{2n}-1} (-1)^{\left[ k \frac{A_{2n}}{B_{2n}} \right]} = 0,$$

so that our proof will be complete if we can show that

$$(5.5) \quad [k\alpha] = \left[ k \frac{A_{2n}}{B_{2n}} \right] \text{ for } 1 \leq k \leq B_{2n} - 1.$$

We proceed by contradiction.

If (5.5) is not true then (note that  $\frac{A_{2n}}{B_{2n}} < \alpha$ )

$$(5.6) \quad k \frac{A_{2n}}{B_{2n}} < m < k\alpha \text{ for some } m \in \mathbb{N}.$$

Hence

$$(5.7) \quad \frac{1}{B_{2n}} \leq m - k \frac{A_{2n}}{B_{2n}} < k\alpha - k \frac{A_{2n}}{B_{2n}} = k \left( \alpha - \frac{A_{2n}}{B_{2n}} \right) < (B_{2n} - 1) \cdot \frac{1}{B_{2n}^2} < \frac{1}{B_{2n}}.$$

This contradiction completes the proof.

COROLLARY. For every irrational  $\alpha \in \mathbb{N}$  we have that

$$(5.8) \quad S_N(\alpha) = 0 \text{ for infinitely many } N \in \mathbb{N}.$$

ADDENDUM.

During the preparation of this note J. VAN DE LUNE and H.J.J. TE RIELE proved the following (more general)

THEOREM. If  $\alpha$  is irrational then  $S_n(\alpha) = 0$  for infinitely many  $n \in \mathbb{N}$ .

REMARK: From now on all fractions  $\frac{p}{q}$  are assumed to be irreducible.

In order to prove the theorem we use the following

LEMMA. If  $p$  is odd then  $S_{2q}(\frac{p}{q}) = 0$ .

PROOF: Observe that the numbers

$$\left[ r \frac{p}{q} \right] \text{ and } \left[ (q+r) \frac{p}{q} \right], \quad 1 \leq r \leq q$$

have different parity.  $\square$

In addition we will use the following well-known

THEOREM (of HURWITZ). If  $\alpha \in \mathbb{R}$  is irrational then there exist infinitely many rationals  $\frac{p}{q}$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2 \sqrt{5}}.$$

PROOF OF THE THEOREM. Let  $H$  be the set of all fractions  $\frac{p}{q}$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2 \sqrt{5}}.$$

It is clear that the proof will be complete if we can show that for every  $\frac{p}{q} \in H$  we have either  $S_{q-1}(\alpha) = 0$  or  $S_{2q}(\alpha) = 0$ .  
We consider a number of cases.

CASE 1.  $\frac{p}{q} \in H$ ,  $p$  even.

Then we have  $S_{q-1}(\alpha) = 0$ .

In order to see this it is clearly sufficient to prove that  $S_{q-1}(\alpha) = S_{q-1}(\frac{p}{q})$ .

Hence it is sufficient to show that

$$[k\alpha] = [k\frac{p}{q}] \text{ for } 1 \leq k \leq q-1.$$

Assuming this does not hold true we have for some  $k$ ,  $1 \leq k \leq q-1$ , that there exists an  $m \in \mathbb{Z}$  such that either

$$k\frac{p}{q} \leq m < k\alpha \text{ (in case } \frac{p}{q} < \alpha)$$

or

$$k\alpha < m \leq k\frac{p}{q} \text{ (in case } \frac{p}{q} > \alpha).$$

Since  $1 \leq k \leq q-1$ , equality in the above cases is impossible and thus

$$\frac{1}{q} \leq |m - k\frac{p}{q}| < k |\alpha - \frac{p}{q}| < \frac{q-1}{q^2\sqrt{5}} < \frac{1}{q},$$

which is a contradiction.

CASE 2.  $\frac{p}{q} \in H$ ,  $p$  odd.

In this case we have  $S_{2q}(\alpha) = 0$ . In order to see this we need only show that  $S_{2q}(\alpha) = S_{2q}(\frac{p}{q})$ .

CASE 2.1.  $\frac{p}{q} < \alpha$ ,  $p$  odd.

It suffices to show that

$$[k\alpha] = [k \frac{p}{q}] \text{ for } 1 \leq k \leq 2q.$$

Since this may be established similarly as in case 1 we consider

CASE 2.2.  $\frac{p}{q} > \alpha$ ,  $p$  odd.

We observe that

$$[q \cdot \frac{p}{q}] = p, [2q \frac{p}{q}] = 2p$$

and

$$[q\alpha] = p-1, [2q\alpha] = 2p-1$$

so that (since  $p$  is odd) it suffices to show that

$$[k\alpha] = [k \frac{p}{q}] \text{ for } 1 \leq k \leq 2q, k \neq q, k \neq 2q.$$

Since this may be shown similarly as before the proof is complete.  $\square$

REMARK. From the above considerations it is easily seen that

(i) if  $p$  is *even* then  $S_{nq}(\frac{p}{q}) = n$  for all  $n \in \mathbb{N}$ .

(ii) if  $p$  is *odd* then  $S_{2nq}(\frac{p}{q}) = 0$  for all  $n \in \mathbb{N}$ .



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